NEW LOCAL SYMMETRY FOR QED IN TWO DIMENSIONS

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Abstract: A new local, covariant and nilpotent symmetry is shown to exist for the interacting BRST invariant $U(1)$ gauge theory in two dimensions of space-time. Under this new symmetry, it is the gauge-fixing term that remains invariant and the corresponding transformations on the Dirac fields turn out to be the analogue of chiral transformations. The extended BRST algebra is derived for the generators of all the underlying symmetries, present in the theory. This algebra turns out to be the analogue of the algebra obeyed by the de Rham cohomology operators of differential geometry. Possible interpretations and implications of this symmetry are pointed out in the context of BRST cohomology and Hodge decomposition theorem.

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The importance of symmetries in physics has gone beyond the mere requirement of an aesthetic appeal to the sophistication of explaining some of the landmark experiments. One such important symmetry in theoretical physics is the gauge symmetry which has turned out to provide the corner-stone for the modern developments in the ideas of unification. The main characteristic feature of theories, based on these symmetries, is the fact that they are endowed with the first-class constraints (in the language of Dirac) [1-3]. A cardinal example of such a class of theories is the quantum electrodynamics (QED) which represents a dynamically closed system of photon ($U(1)$ gauge field A_μ) and electrically charged particles (e.g. electrons and positrons which are described by the Dirac fields ψ and $\bar\psi$). For the covariant canonical quantization of these gauge theories, the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [4,5] turns out to be quite handy. In this formalism, unitarity and gauge invariance are respected together at any arbitrary order of perturbation theory. The purpose of the present letter is to shed some light on the existence of some local, covariant and continuous symmetries which have not been explored hitherto in the context of two dimensional (2D) QED and to establish that this *interacting* theory provides *a physically tractable field theoretical model for the Hodge theory.* The generators for the underlying symmetries of the theory obey an algebra that is reminiscent of the algebra of de Rham cohomology operators of differential geometry defined on a compact manifold.

Let us begin with a D-dimensional BRST invariant Lagrangian density ($\mathcal{L}_B$) for the interacting $U(1)$ gauge theory in the Feynman gauge:

$$\mathcal{L}_B = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar\psi (i\gamma^\mu \partial_\mu - m)\psi - e\bar\psi\gamma^\mu A_\mu \psi + B(\partial\cdot A) + \tfrac{1}{2} B^2 - i\partial_\mu \bar C \partial^\mu C, \qquad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, B is the Nakanishi-Lautrup auxiliary field, $(\bar C)C$ are the Faddeev-Popov (anti)ghost fields ($\bar C^2 = C^2 = 0$) and indices $\mu, \nu = 0, 1, 2, ...D-1$ represent the flat Minkowski space-time directions. It can be checked that the above Lagrangian density remains quasi-invariant ($\delta_B \mathcal{L}_B = \eta \partial_\mu [B \partial^\mu C]$) under the following off-shell nilpotent ($\delta_B^2 = 0$) BRST transformations:

$$\begin{array}{lllll} \delta_B A_\mu & = & \eta\, \partial_\mu\, C, \quad \delta_B C = 0, \quad \delta_B \bar C = i\,\eta\, B, \quad \delta_B B = 0, \\ \delta_B \psi & = & -i\eta e C\psi, \quad \delta_B \bar\psi = i\eta e C\bar\psi, \quad \delta_B(\partial\cdot A) = \eta\Box C, \quad \delta_B F_{\mu\nu} = 0, \end{array} \qquad (2)$$

where η is an anticommuting ($\eta C = -C\eta, \eta\bar C = -\bar C\eta, \eta\psi = -\psi\eta, \eta\bar\psi = -\bar\psi\eta$) space-time independent transformation parameter. Using Noether theorem, it is straightforward to check that the generator for the above transformations is the nilpotent ($Q_B^2 = 0$) BRST charge

$$\begin{array}{lll} Q_B & = & \int d^{(D-1)}x\, [\, \partial_i F^{0i} C + B\dot C - e\bar\psi\gamma_0 C\psi\,], \\ & \equiv & \int d^{(D-1)}x\, [\, B\dot C - \dot B C\,], \end{array} \qquad (3)$$

where the latter expression for Q_B has been obtained by exploiting the equation of motion ($\partial_\mu F^{\mu\nu} - \partial^\nu B = e\bar\psi\gamma^\nu\psi$). The global scale invariance of (1) under $C \to e^{-\lambda} C, \bar C \to e^{\lambda}\bar C, A_\mu \to A_\mu, B \to B$ (where λ is a global parameter), leads to the derivation of a

conserved ghost charge (Q_g)

$$Q_g = -i \int d^{(D-1)}x \; [\; C\dot{\bar{C}} + \bar{C}\dot{C} \;]. \tag{4}$$

Together, these conserved charges satisfy the following algebra

$$\begin{array}{lllll} Q_B^2 & = & \frac{1}{2}\{Q_B, Q_B\} = 0, & & i[Q_g, Q_B] = +Q_B, \\ Q_{AB}^2 & = & \frac{1}{2}\{Q_{AB}, Q_{AB}\} = 0, & & i[Q_g, Q_{AB}] = -Q_{AB}, \end{array} \tag{5}$$

where Q_{AB} is the anti-BRST charge which can be readily obtained from (3) by the replacement :$C \rightarrow i\bar{C}$ [6-8]. Note that $C \rightarrow \pm i\bar{C}, \bar{C} \rightarrow \pm iC$ is the symmetry of the ghost action ($I_{F.P.} = -i \int d^D x \; \partial_\mu \bar{C} \partial^\mu C$) in any arbitrary dimension of space-time.

In the past few years, many authors [9-12] have attempted to explore the possibilities of obtaining new BRST-type symmetries for the Lagrangian density (1) of QED in the hope of establishing a deeper connection with the mathematical aspects of BRST cohomology in any arbitrary dimension of space-time (see, e.g., Refs. [9,12]). However, the symmetry transformations turn out to be nonlocal and noncovariant. In the relativistic covariant formulation of these symmetries [13] the manifest nilpotency is lost and it is restored only under certain specific restrictions on the parameters of the theory. The central theme of our present short note is to show that in two dimensions of space-time there exists a local, continuous, covariant and nilpotent BRST-type symmetry under which the gauge-fixing term of the Lagrangian density (1) remains invariant and corresponding transformations on Dirac fields turn out to be the analogue of chiral transformations. This symmetry transformation is not the generalization of the D-dimensional symmetries [9-12] explored by others to two dimensions of space-time. *Rather, it is a new symmetry in its own right.* Contrary to the above symmetry transformation, it is the Abelian field strength tensor (two-form) that remains invariant under the usual BRST symmetry and the Dirac fields transform as the analogue of gauge transformations (see, e.g., eqn.(2)). We christen the new symmetry as the dual-BRST symmetry because the gauge-fixing term is Hodge dual to the field strength tensor (two-form) of Abelian $U(1)$ gauge theory in any arbitrary dimension of space-time † [14,15]. This duality is also reflected at the level of transformations for the Dirac fields where the analogues of gauge- and chiral transformations are dual to each-other.

In two $(1+1)$ dimensions of space-time, there exists only one component (i.e. electric field $E = F_{01}$) of the field strength tensor $F_{\mu\nu}$. Thus, the analogue of the BRST invariant Lagrangian density (1) is:

$$\mathcal{L}_B = \tfrac{1}{2}\; E^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial \cdot A) + \tfrac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C, \tag{6a}$$

which can be recast as:

$$\mathcal{L}_{\mathcal{B}} = \mathcal{B}E - \tfrac{1}{2}\; \mathcal{B}^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial \cdot A) + \tfrac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C, \tag{6b}$$

†The vector potential A_μ of the $U(1)$ gauge theory is defined through one-form $A = A_\mu dx^\mu$. The gauge-fixing term $\partial \cdot A = \delta A$ is Hodge dual to the two-form $F = dA$ where $\delta = \pm * d *$ is the adjoint(dual) exterior derivative and d is the exterior derivative (see, e.g., Ref. [15]).

by introducing another auxiliary field $\mathcal{B}$. It can be checked that under the following off-shell nilpotent ($\delta_D^2 = 0$) dual-BRST transformations

$$\begin{array}{lcl} \delta_D A_\mu &=& -\eta \varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad \delta_D \bar{C} = 0, \quad \delta_D C = -i\eta \mathcal{B}, \quad \delta_D \mathcal{B} = 0, \quad \delta_D B = 0, \\ \delta_D \psi &=& -i\eta e \bar{C} \gamma_5 \psi, \quad \delta_D \bar{\psi} = i\eta e \bar{C} \gamma_5 \bar{\psi}, \quad \delta_D (\partial \cdot A) = 0, \quad \delta_D E = \eta \Box \bar{C}, \end{array} \tag{7}$$

the Lagrangian density (6b) (with $m = 0$) transforms as: $\delta_D \mathcal{L}_{\mathcal{B}} = \eta \partial_\mu (\mathcal{B} \partial^\mu \bar{C})$ ‡. Exploiting the Noether theorem, it can be checked that the above transformations are generated by

$$\begin{array}{lcl} Q_D &=& \int dx \; [\; \mathcal{B} \dot{\bar{C}} + e \bar{\psi} \gamma_1 \bar{C} \psi - (\partial_1 B) \bar{C} \;], \\ &\equiv& \int dx \; [\; \mathcal{B} \dot{\bar{C}} - \dot{\mathcal{B}} \bar{C} \;], \end{array} \tag{8}$$

where the latter expression for Q_D has been obtained by using the equation of motion ($\varepsilon^{\mu\nu} \partial_\nu \mathcal{B} + \partial^\mu B = -e \bar{\psi} \gamma^\mu \psi$) for the photon field, present in the Lagrangian density (6b). Using the following BRST quantization conditions (with $\hbar = c = 1$):

$$\begin{array}{l} [A_0(x,t), B(y,t)] = i\delta(x-y), \\ [A_1(x,t), \mathcal{B}(y,t)] = i\delta(x-y), \\ \{\psi(x,t), \psi^\dagger(y,t)\} = -\delta(x-y), \\ \{C(x,t), \dot{\bar{C}}(y,t)\} = \delta(x-y), \\ \{\bar{C}(x,t), \dot{C}(y,t)\} = -\delta(x-y), \end{array} \tag{9}$$

(and rest of the (anti)commutators are zero), it can be seen that Q_D is indeed the generator for the transformations (7) if we exploit the following relationship

$$\delta_D \Phi = -i\eta [\Phi, Q_D]_\pm, \tag{10}$$

where $[\ ,\ ]_\pm$ stands for (anti)commutator for the generic field Φ being (fermionic)bosonic in nature. It is straightforward to check that $Q_D^2 = \frac{1}{2}\{Q_D, Q_D\} = 0$ due to (9). A simpler way to see this fact is: $\delta_D Q_D = -i\eta \{Q_D, Q_D\} = 0$ by exploiting (7) and (8).

It is very natural to expect that the anticommutator of these two transformations ($\{\delta_B, \delta_D\} = \delta_W$) would also be the symmetry transformation (δ_W) for the Lagrangian density (6b) (with $m = 0$). This is indeed the case as can be seen that under the following bosonic ($\kappa = -i\eta\eta'$) transformations corresponding to δ_W

$$\begin{array}{lcl} \delta_W A_\mu &=& \kappa (\partial_\mu \mathcal{B} + \varepsilon_{\mu\nu} \partial^\nu B), \qquad \delta_W \mathcal{B} = 0, \qquad \delta_W B = 0, \\ \delta_W (\partial \cdot A) &=& \kappa \Box \mathcal{B}, \qquad \delta_W E = -\kappa \Box B, \quad \delta_W C = 0, \quad \delta_W \bar{C} = 0, \\ \delta_W \psi &=& \kappa i e (\gamma_5 B - \mathcal{B}) \psi, \qquad \delta_W \bar{\psi} = -\kappa i e (\gamma_5 B - \mathcal{B}) \bar{\psi}, \end{array} \tag{11}$$

the Lagrangian density (6b) (with $m = 0$) transforms as: $\delta_W \mathcal{L}_{\mathcal{B}} = \kappa \partial_\mu [B \partial^\mu \mathcal{B} - \mathcal{B} \partial^\mu B]$. Here η and η' are the transformation parameters corresponding to the transformations δ_B

‡ We adopt the notations in which the flat 2D Minkowski metric $\eta_{\mu\nu} = \text{diag}\,(+1, -1), \gamma^0 = \sigma_2, \gamma^1 = i\sigma_1, \gamma_5 = \gamma^0 \gamma^1 = \sigma_3, \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \gamma_\mu \gamma_5 = \varepsilon_{\mu\nu} \gamma^\nu, \varepsilon_{01} = \varepsilon^{10} = +1, F_{01} = \partial_0 A_1 - \partial_1 A_0 = E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu = F^{10}, \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_1)^2$ and here $\sigma's$ are the usual 2×2 Pauli matrices.

and δ_D respectively. The generator for the above transformations is:

$$\begin{aligned} W &= \int dx\; [\; B(\partial_1 B + e\bar\psi\gamma_1\psi) - \mathcal{B}(\partial_1\mathcal{B} - e\bar\psi\gamma_0\psi)\;], \\ &\equiv \int dx\; [\; B\dot{\mathcal{B}} - \dot{B}\mathcal{B}\;], \end{aligned} \tag{12}$$

where the latter expression for W has been obtained due to the use of equation of motion: $\varepsilon^{\mu\nu}\partial_\nu\mathcal{B}+\partial^\mu B = -e\bar\psi\gamma^\mu\psi$. There are other simpler ways to derive the expression for W. For instance, it can be seen that anticommutator of Q_B and Q_D leads to the derivation of W (i.e. $\{Q_B, Q_D\} = W$) if we exploit the basic (anti)commutators of equation (9). Furthermore, since Q_B and Q_D are generators for the transformations (3) and (7) respectively, it can be seen that the following relationships

$$\begin{aligned} \delta_D Q_B &= -i\eta\{Q_B, Q_D\} = -i\eta W, \\ \delta_B Q_D &= -i\eta\{Q_D, Q_B\} = -i\eta W, \end{aligned} \tag{13}$$

lead to the definition and derivation of W. Together, all the above generators obey the following extended BRST algebra

$$\begin{array}{l} [W, Q_k] = 0, \qquad\qquad k = g, B, D, AB, AD, \\ Q_B^2 = Q_D^2 = Q_{AB}^2 = Q_{AD}^2 = 0, \qquad \{Q_D, Q_{AD}\} = 0, \\ \{Q_B, Q_D\} = \{Q_{AB}, Q_{AD}\} = W, \qquad \{Q_B, Q_{AB}\} = 0, \\ i[Q_g, Q_B] = Q_B, \qquad i[Q_g, Q_{AB}] = -Q_{AB}, \\ i[Q_g, Q_D] = -Q_D, \qquad i[Q_g, Q_{AD}] = Q_{AD}, \end{array} \tag{14}$$

and the rest of the (anti)commutators are zero. Here Q_{AD} is the anti-dual BRST charge which can be readily derived from (8) by the replacement :$\bar{C} \to \pm iC$. It is evident that the operator W is the Casimir operator for the whole algebra. The *mathematical aspects* of the representation theory for the above kind of BRST algebra have been discussed in Refs. [16-18]. It will be noticed that the ghost number for Q_B and Q_{AD} is $+1$ and that of Q_D and Q_{AB} is -1. Now, given a state $|\phi>$ with ghost number n in the quantum Hilbert space (i.e. $iQ_g|\phi>= n|\phi>$), it can be readily seen, using the algebra (14), that

$$\begin{aligned} iQ_gQ_B|\phi> &= (n+1)Q_B|\phi>, \qquad iQ_gQ_{AD}|\phi>= (n+1)Q_{AD}|\phi>, \\ iQ_gQ_D|\phi> &= (n-1)Q_D|\phi>, \qquad iQ_gQ_{AB}|\phi>= (n-1)Q_{AB}|\phi>, \\ iQ_gW\;|\phi> &= n\;W\;|\phi>. \end{aligned} \tag{15}$$

This shows that the ghost numbers for the states $Q_B|\phi>$ (or $Q_{AD}|\phi>$), $Q_D|\phi>$ (or $Q_{AB}|\phi>$) and $W|\phi>$ in the quantum Hilbert space are $(n+1), (n-1)$ and n respectively.

At this stage, it is worth mentioning that in Refs. [19-21], the analogous expressions for Q_B, Q_D, W (cf. eqns.(3),(8),(12)) have been derived for the *free* 2D Abelian- and non-Abelian gauge theories (having no interaction with matter fields). Recently, these results have also been shown to exist for the *free* Abelian two-form gauge theory in $(3+1)$ dimensions of space-time [22]. The topological properties of these *free* 2D theories have been shown to be encoded in the vanishing of the operator W when equations of motion are

exploited. On the contrary, as it turns out, the operator W is defined off-shell as well as on-shell for the 2D *interacting* BRST invariant $U(1)$ gauge theory. This is because of the fact that even though $U(1)$ gauge field is topological (i.e. without any propagating degrees of freedom), it is coupled to the Dirac fields here and fermionic degrees of freedom are present in the off-shell as well as on-shell expression for W. Thus, the present theory is an example of an *interacting* topological field theory in 2D.

It is interesting to note that the algebra of Q_B, Q_D and W in equation (14) is exactly identical to the corresponding algebra for the exterior derivative ($d,\ d^2 = 0$), dual exterior derivative ($\delta = \pm * d *,\ \delta^2 = 0$) and the Laplacian operator ($\Delta = (d+\delta)^2 = d\delta + \delta d$) in the context of discussion of the de Rham cohomology [14,15]. Furthermore, it can be readily seen that the operation of these generators on a state with ghost number n (cf. eqn.(15)) is same as the operation of the above cohomological operators on a differential form of degree n. Thus, it is clear that the BRST cohomology can be defined comprehensively in terms of the above operators and Hodge decomposition theorem can be expressed cogently in the quantum Hilbert space of states where any arbitrary state $|\phi>_n$ (with ghost number n) can be written as the sum of a harmonic state $|\omega>_n$ ($W|\omega>_n = 0, Q_B|\omega>_n = 0, Q_D|\omega>_n = 0$), a BRST exact state ($Q_B|\theta>_{n-1}$) and a co-BRST exact state ($Q_D|\chi>_{n+1}$). Mathematically, this statement can be expressed by the following equation [§]

$$|\phi>_n = |\omega>_n + Q_B|\theta>_{n-1} + Q_D|\chi>_{n+1}\ . \tag{16}$$

It is obvious, therefore, that the above symmetry generators Q_B, Q_D and W have their counterparts in differential geometry as the de Rham cohomology operators d, δ and Δ respectively [14,15] for the discussion of cohomological aspects of differential forms. It is a peculiarity of the BRST formalism that the above cohomological operators can be also identified with the generators Q_{AD}, Q_{AB} and $W = \{Q_{AB}, Q_{AD}\}$ respectively. Thus, the mapping is: $(Q_B, Q_{AD}) \Leftrightarrow d,\ (Q_D, Q_{AB}) \Leftrightarrow \delta,\ W = \{Q_B, Q_D\} = \{Q_{AD}, Q_{AB}\} \Leftrightarrow \Delta$.

It will be very useful to explore the impact of this new symmetry in the context of symmetries of the Green's functions for QED and derive the analogue of Ward-Takahashi identities. This new symmetry, being connected with the analogue of chiral transformation, is expected to play an important role in throwing some light on the 2D Adler-Bardeen-Jackiw anomaly in the framework of BRST cohomology and Hodge decomposition theorem. Furthermore, this symmetry might turn out to provide a key tool in proving the consistency and unitarity of the anomalous gauge theory in 2D (see, e.g., Refs. [23, 24] and references therein). The generalization of this new symmetry to 2D non-Abelian gauge theory (having local gauge interaction with matter fields) is another future direction that can be pursued. The insights gained in these studies might provide a clue for the generalization of this new symmetry to physical four dimensional gauge theories. These are some of the issues under investigation and a detailed discussion will be reported elsewhere [25].

[§] This equation is the analogue of the Hodge decomposition theorem which states that, on a compact manifold, any arbitrary n-form $f_n (n = 0, 1, 2.....)$ can be written as the sum of a harmonic form ω_n ($\Delta\omega_n = 0, d\omega_n = 0, \delta\omega_n = 0$), an exact form dg_{n-1} and a co-exact form δh_{n+1} as: $f_n = \omega_n + dg_{n-1} + \delta h_{n+1}$.

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